EUCLIDEAN DOMAINS WITH UNIFORMLY ABELIAN LOCAL FUNDAMENTAL GROUPS

BY

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1. **Introduction.** If C and C^1 are homeomorphic closed subsets of the Euclidean three space, R^3, a necessary condition that their complementary sets be homeomorphic is that the fundamental groups of their residual sets be isomorphic. When these groups are not isomorphic the situation may be described roughly by saying that C and C^1 are imbedded differently in R^3 either in the large, or in the small, or both. Examples of the first situation arise if C and C^1 are polygonal simple closed curves, one knotted, the other unknotted. Examples of the second variety arise if C and C^1 are taken to be an ordinary linear interval and an arc of Antoine [2](1) respectively.

In this note we are interested in establishing conditions on the complement of certain classes of homeomorphic closed sets such that the corresponding fundamental groups will vanish. A general problem, which is solved below only in special cases indeed, is: If C is an absolute retract in the n-sphere, S^n, under what conditions does $\pi_1(S^n-C)$ vanish?

In case C is a topological i-cell, $i=1, 2, \cdots, n$, it is sufficient that S^n-C have uniformly abelian local fundamental groups (Theorem IV).

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2. Let X be a topological space, A a subset of X. We shall consider the 1-dimensional homology groups $H_1(A)$ and $H_1(X)$ based on singular homologies with integer coefficients. The identity map $i:A\rightarrow X$ induces a homomorphism $i_*:H_1(A)\rightarrow(H_1X)$. The image of this homomorphism will be denoted by $H_1(A, X)$ and is the homology group obtained by considering 1-cycles in A and bounding in X (not to be confused with the relative homology group $H_1(X, A)$).

Assume that both X and A are arcwise connected and let $p\in A$ be the base point used to define the fundamental groups $\pi_1(A)$ and $\pi_1(X)$. Again the identity map $i:A\rightarrow X$ induces a homomorphism $i_\#:\pi_1(A)\rightarrow\pi_1(X)$. The image of this homomorphism will be denoted by $\pi_1(A, X)$. Up to an isomorphism, this group is independent of the choice of $p\in A$.

Clearly $H_1(A, X)$ and $\pi_1(A, X)$ are subgroups of $H_1(X)$ and $\pi_1(X)$, respectively. If $B\subset A$, then

$$H_1(B, X) \subset H_1(A, X) \quad \text{and} \quad \pi_1(B, X) \subset \pi_1(A, X).$$

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(1) The numbers in brackets refer to the bibliography at the end of the paper.

(For the second inclusion, assume B is arcwise connected and $p \in B$.)

Consider the diagram

$$\begin{array}{ccc} \pi_1(A) & \xrightarrow{i_\#} & \pi_1(X) \\ \nu_1 \downarrow & & \downarrow \nu_2 \\ H_1(A) & \xrightarrow[i_*]{} & H_1(X) \end{array}$$

where ν_1, ν_2 are the natural homomorphisms of the fundamental group *onto* the homology group. The kernels of ν_1 and ν_2 are the commutator groups $[\pi_1(A), \pi_1(A)]$ and $[\pi_1(X), \pi_1(X)]$. Since $i_*\nu_1 = \nu_2 i_\#$, it follows that

$$\begin{aligned} H_1(A, X) &= \text{image } i_* = \text{image } (i_*\nu_1) \\ &= \text{image } (\nu_2 i_\#) = \nu_2(\text{image } i_\#) \\ &= \nu_2[\pi_1(A, X)]. \end{aligned}$$

Thus

$$\nu_2\colon\ \pi_1(A, X) \to H_1(A, X)$$

is a mapping onto. The kernel of this homomorphism is the group

$$\mathcal{C}(A, X) = \pi_1(A, X) \cap [\pi_1(X), \pi_1(X)]$$

so that

$$(1) \qquad \pi_1(A, X)/\mathcal{C}(A, X) \approx H_1(A, X)\text{(2)}.$$

The group $\mathcal{C}(A, X)$ can be defined directly by considering closed paths in A (about p) which are homotopic in X to commutators of paths in X; the homotopy of paths in A is considered in X. In particular $\mathcal{C}(A, X) = 0$ means that every closed path in A which is homotopic (in X) to the commutator of two paths in X is null-homotopic in X.

In the above discussion the hypothesis that X be arcwise connected can be disposed of as long as A is arcwise connected. Indeed, only the arc-component of X containing A comes into play. Thus the definitions of $\pi_1(A, X)$, $C(A, X)$ and formula (1) remain valid with only A arcwise connected.

Observe further that the statement $\pi_1(A, X) = 0$ (or $\mathcal{C}(A, X) = 0$) may be interpreted, even if A is not arcwise connected, to mean that $\pi_1(D, X) = 0$ (or $\mathcal{C}(D, X) = 0$) for every arcwise connected component D of A.

3. Let X be a topological space, A an open subset of X. A set $U \subset A$ will be called a neighborhood of $p \in X$ in A if $U = A \cap V$ for some open set V of X containing p.

The set A will be called k-lc at the point $p \in X$ if for every neighborhood U

(2) The symbol $\approx$ is read "is isomorphic to," $\simeq$ "is homotopic to," and $\sim$ "is homologous to."

of p in A there is a neighborhood V of p in A such that every (singular integral) k-cycle in V bounds in U (that is, if $H_k(V, U)=0$ where $H_k(V, U)$ is the image of $H_k(V)\to H_k(U)$). If A is k-lc at every $p\in A$ then A is called k-lc; if A is k-lc at every $p\in X$ then A is k-ulc. If A is k-lc (or k-ulc) for each $k=0, 1, \cdots, n$ then A is lc^n (or ulc^n).

The set A will be called k-LC at $p\in X$ if for every neighborhood U of p in A there is a neighborhood V of p in A such that every singular k-sphere in V is null-homotopic in U (that is, $\pi_k(V, U)=0$). If A is k-LC at every $p\in A$ then A is k-LC; if A is k-LC at every $p\in X$ then A is k-UCL. If A is k-LC (or k-ULC) for each $k=0, 1, \cdots, n$ then A is LC^n (or ULC^n). If $A=X$, then LC^n and ULC^n coincide. It is immediate that ulc^0 and ULC^0 agree.

The open set A in the space X will be said to have *uniformly abelian local fundamental groups* (abbreviate un.a.l.g.) provided for each $p\in X$ and each neighborhood U of p in A there is a neighborhood V of p such that $V\subset U$ and $\mathcal{C}(V, U)=0$.

THEOREM I. *Let A be an open subset of X. Then A is 1-ULC if and only if A is 1-ulc and has the un.a.l.g. property.*

Proof. Suppose A is 1-ULC. Let U be a neighborhood in A of $p\in X$. There is a neighborhood V of p in A such that $V\subset U$ and $\pi_1(V, U)=0$. Let W be any arcwise connected component of V. Then $\mathcal{C}(W, U)\subset\pi_1(W, U)$ and $H_1(W, U)\approx\pi_1(W, U)/\mathcal{C}(W, U)$. Thus $\mathcal{C}(W, U)=0$ and $H_1(W, U)=0$ and consequently $\mathcal{C}(V, U)=0$ and $H_1(V, U)=0$. Hence A is un.a.l.g. and 1-ulc. Conversely, assume that A is un.a.l.g. and 1-ulc. Let U be a neighborhood in A of a point $p\in X$. We can select neighborhoods V_1, V_2 of p in A such that $V_1\subset U$, $H_1(V_1, U)=0$, $V_2\subset U$, $\mathcal{C}(V_2, U)=0$. Let W be an arcwise connected component of $V_1\cap V_2$. Then $H_1(W, U)=0$ and $\mathcal{C}(W, U)=0$. Since $\pi_1(W, U)/\mathcal{C}(W, U)\approx H_1(W, U)$ it follows that $\pi_1(W, U)$, thus $\pi_1(V, U),\approx 0$ and A is 1-ULC.

THEOREM II. *Let M be a closed, generalized, $n-1$ manifold in the n-dimensional sphere, S^n, and A one of the components of S^n-M. Then A is ULC^1 if and only if A is un.a.l.g.*

Proof. Wilder [7] has proved that A is ulc^1 (even ulc^n). Thus Theorem II follows from Theorem I. (In Wilder's definition of lc^n and ulc^n, Vietoris' cycles were used; however, since A is an open subset of S^n this will give the same concept as the singular theory.)

THEOREM III. *Let M be a topological $(n-1)$-sphere in S^n and A one of the components of S^n-M. If A is un.a.l.g., then A and $\overline{A}$ are contractible.*

Proof. Theorem II implies that A is ULC^1. Thus Theorem III follows from Theorem 10 of [3].

4. THEOREM IV. *Let C be a closed, topological i-cell, $i=1, 2, \cdots, n$, in*

the euclidean n-sphere, S^n *such that* $S^n - C$ *has uniformly abelian local fundamental groups. Then* $\pi_1(S^n - C) = 0$.

4.1. **Proof.** $i = n$. Let M be the boundary of C and set $A = S^n - C$. Then Theorem III applies.

4.2. $i = n-1$. As above put $A = S^n - C$. Now A fails to be 0-ULC as is seen by taking points near together on opposite side of C to form a 0-sphere. (Thus Theorem 8a of the paper referred to above cannot be applied directly.)

4.3. Observe that A is 1-ulc over the integers [5; 7]. If we combine this with the fact that A has uniformly abelian local fundamental groups, Theorem I asserts A is 1-ULC.

4.4. The next step is to define a set $X = H \cup M$. The set H will be homeomorphic to A and in addition will be ULC^1. The set M will be the "boundary" of H. The additional properties needed of M are that it is LC^1 and $\pi_1(M) = 0$. Finally, we shall show $\pi_1(H, X) = 0$. Granting these preliminary results. Theorem 8 of [3] applies (where A of that theorem corresponds to our H). Hence $\pi_1(H) = 0$. Since H is constructed homeomorphic to A above, $\pi_1(A) = 0$.

4.5. Lemma. *If M is a compact, LC^0 subset of a space X and X is LC^1 at every point of M, then every closed path in X sufficiently near M can be homotopically deformed into a closed path in M.*

Let the components of M be $M_1, \cdots, M_k$. Define $0 < 3\epsilon <$ min distance $M_i, M_j, i \neq j$. Since X is 1-LC at the points of M there is a $0 < \delta' < \epsilon$ such that any closed path in(3) $S(M, \delta')$ of diameter $< \delta'$ is $\simeq 0$ in X by a deformation moving each point $< \epsilon$. To δ' there is a δ^0 such that if $p, q \in M$, distance $p, q < \delta^0$, there is an arc in M joining these points of diameter $< \delta'/3$. Similarly, there is an δ^{00} such that if $p, q \in S(M, \delta')$, distance $p, q < \delta^{00}$, there is an arc in X joining these points of diameter $< \delta'/3$. Let $3\delta = \min \delta^0, \delta^{00}$. Define $V = S(M, \delta)$.

Let $f(S^1)$ be any closed path in $V_M = S(M, \delta) - M$. Subdivide $f(S^1)$ into consecutive (singular) arcs of diameter $< \delta$. Let the vertices be $x_0, \cdots, x_m (= x_0)$. To x_i associate $y_i \in M$ such that distance $x_i, y_i < \delta$. Since $\delta \leqq \delta^{00}$, there is an arc J_i in X joining these points of diameter $< \delta'/3$. Since $3\delta \leqq \delta^0$, there is an arc $y_i y_{i+1}$ joining y_i and y_{i+1} in M of diameter $< \delta'/3$. Then $x_i x_{i+1} \cup J_i \cup J_{i+1} \cup y_i y_{i+1}$ is a singular 1-sphere in $S(M, \delta')$ of diameter $< \delta'$. Hence it is $\simeq 0$ by a deformation moving no point as much as ϵ. Piecing together the singular disks which realize these deformations in such a way as to preserve continuity, we have $f(S^1) \simeq g(S^1) = y^0 y_1 \cup \cdots \cup y_{m-1} y_0 \subset M$.

4.6. *Definition of H.*

4.61. Let c be any interior point of C. Let l be an arc having c as an end point and $l - c \subset S^n - C$. Since $q^0(c, S^n - C) = 1$ [1, ¶2, corollary II], there exist $\epsilon > \delta > 0$ such that $S(c, \delta) - C = U \cup V$, where U and V are mutually

(3) $S(X, \delta)$ means all points in space at a distance from X less than δ. $F(X, \delta)$ means all points in space at a distance from X equal to δ.

separated and any 0-cycle in U (or in V) is ~ 0 in $S(c, \epsilon)-C$. Define $0<3\epsilon'<\epsilon-\delta$. Then if $p\in U\cap S(c, \epsilon')$ and $q\in V\cap S(c, \epsilon')$, any arc in S^n-C from p to q must have a diameter $>2\epsilon'$ since it must meet both(3) $F(c, \epsilon')$ and $F(c, 3\epsilon')$. Thus if we choose a point d on l near enough to c, the subarc of l, dc, will have a diameter $<\epsilon'$ and any arc from d to a point d^1 near c but on the other side of C from d will have a diameter $>2\epsilon'$.

4.62. A new metric, ρ, the "relative distance metric" is defined(4) on S^n-C by $\rho(p\ q)=$glb diameter of all connected sets $J\supset p\cup q$, $J\subset S^n-C$. Define $w(p)=\rho(p, d)$. This is a real-valued, continuous function on S^n-C and if $c^1\in C$ is sufficiently near c, then w', $w''=$respectively, glb, lub $w(p)$, $p\to c^1$, $p\in S^n-C$ are distinct numbers (4.61).

Intuitively, it would seem that w', w'' are distinct for any interior point $c^1\in C$. However, we are unable to prove this, hence the construction below.

4.63. Let $(c_1, c_2, \cdots)$ be dense in C, $c_i\in$interior C. For each c_i choose an arc l_i as above (4.3) and a point d_i such that diameter $c_id_i<\epsilon_i$ and any arc T from d_i to a point d_i' near c_i but on the other side of C from d_i, $T\subset S^n-C$, implies diameter $T>2\epsilon_i$.

4.64. Define a sequence of real-valued, continuous functions on S^n-C by $w_i(p)=\rho(p, d_i)$. Define a subset H of a Hilbert cube as follows:

If $(x_1, x_2, \cdots, x_n)=p\in S^n-C$, $(x_1, x_2, \cdots, x_n)\leftrightarrow(x_1, x_2, \cdots, x_n, w_1, w_2, w_3, \cdots)$. If $q=(x_1', x_2', \cdots, x_n')$ is a second point of S^n-C, we may define distance in H by

$$\mu(p, q) = \sum_1^n | x_i - x_i' | + \sum_1^\infty \frac{| w_i(p) - w_i(q) |}{2^i}.$$

4.7. *Proof that H is* ULC[1]. The mapping h defined on H by setting each $w_i=0$ is a homeomorphism, $h(H)=S^n-C$. In fact, h clearly diminishes the diameters of subsets of H. On the other hand, from the definition of ρ (and hence w_i), the homeomorphism h^{-1} carries *connected* subsets of S^n-C of diameter $<\epsilon$ into subsets of H of diameter $<3\epsilon$. By (4.3), $A=S^n-C$ is 1-ULC, hence, by the last sentence, H is 1-ULC.

4.71. To prove that H is 0-ULC, let $\phi(p, q)=\sum_1^n |x_i-x_i'|$, where $p=(x_1, \cdots, x_n)$ and $q=(x_1', \cdots, x_n')$. Let (u_k, v_k), $k=1, 2, \cdots$, be pairs of points in H such that lim $\mu(u_k, v_k)=0$. Since $\phi[h(u_k), h(v_k)]\leq\mu(u_k, v_k)$, $\lim_{k\to\infty}\phi[h(u_k), h(v_k)]=0$. Suppose $h(u_k)\to\gamma\leftarrow h(v_k)$, $\gamma\in C$. If $\gamma\in$ interior of C and for all large k, $h(u_k)$, $h(v_k)$ lie on the same side of C as explained above (4.61), there exists connected subsets T_k of A of small diameter joining $h(u_k)$ and $h(v_k)$. But h^{-1} carries small connected sets in A into small connected sets in H, that is, the requirement for 0-ULC is in this case fulfilled. If for some subsequence $h(u_{i_k})$, $h(v_{i_k})$ lie on opposite sides of C near γ, then $w_j\{h(u_{i_k})\}$

(4) See [6] for properties of this metric and references to the work of S. Mazurkiewicz, who introduced this metric originally.

$-w_j\{h(v_{i_k})\} \geqq 2\epsilon_j > 0$, hence $\mu(u_{i_k}, v_{i_k}) \geqq \epsilon_j/2^j$, denying the hypothesis that $\lim_{k\to\infty} u(u_k, v_k) = 0$. Thus H is 0-ULC.

4.72. The above remarks show that if $p \in$ interior of C and if w_j', w_j'' = glb, lub, respectively, of $w_j(q)$, $q \to p$, $q \in S^n - C$, then $w_j' \neq w_j''$ for *some* j.

4.8. *Definition of M and X*. Define M' as the totality of points $(x_1, \cdots, x_n, w_1', w_2', \cdots)$ in a Hilbert cube, where w_i' is glb $w_i(\bar{p})$, $\bar{p} \to (x_1, \cdots, x_n) \in C$, $\bar{p} \in S^n - C$. Then M' is a 1–1 continuous image of C, that is, a topological $n-1$ cell. That M' is a 1–1 map of C under this association is clear. To prove it is continuous let $p = (x_1, \cdots, x_n)$ and $p' = f(p)$ its image in M'. Observe that if $C \subset S(p, d)$, then each $w_i(q)$, $q \in S(p, d)$, $i = 1, 2, \cdots$, is $<$ some constant λ. In fact the same constant λ may be used for all $p \in C$.

Let $\epsilon > 0$. Choose N such that $\sum_N^\infty \lambda/2^i < \epsilon/6$. To $\epsilon/12$ there is a $\delta > 0$ such that $S(p, \delta) - C = U \cup V$, where U and V are mutually separated if p is interior to C (and $U = V$ otherwise) and any 0-cycle in U (or in V) is ~ 0 in $S(p, \epsilon/12) - C$. Now let $q = (x_1, \cdots, x_n) \in C \cap S(p, \delta)$. Then $w_i'(q) < w_i'(p) + \epsilon/6$ and $w_i'(p) < w_i(q)' + \epsilon/6$, that is, $|w_i'(p) - w_i'(q)| < \epsilon/6$. Hence $\mu[f(p), f(q)] < \sum_1^n |x_i - x_i'| + \epsilon/6(1 + 1/2 + 1/4 + \cdots + 1/2^{N-1}) + \sum_{N-1}^\infty \lambda/2^i < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$.

Define M$''$ as the totality of points $(x_1, \cdots, x_n, w_1'', w_2'', \cdots)$, where w_i'' is the lub $w_i(p)$, $p \to (x_1, \cdots, x_n) \in C$, $p \in S^n - C$. Likewise, M'' is a topological $n-1$ cell.

Define $X = H \cup M' \cup M''$. Again, let h denote the continuous mapping which results by setting each $w_i = 0$, this time h being defined on X. Then $h(X) = S^n$. Let J be the $n-2$ sphere that is the boundary of C. If $p \in C - J$, $h^{-1}(p)$ consists of two points, one in M', the other in M'' (4.72). If $p \in J$, $h^{-1}(p)$ is a single point (since $q^0(p, S^n - C) = 0$, each $w_j' = w_j''$). Now h is continuous and 1–1 on the compact set $h^{-1}(J)$. Thus J and $h^{-1}(J)$ are homeomorphic. Hence $h^{-1}(J) = M' \cap M''$ is a topological $n-2$ sphere. We conclude $M = M' \cup M''$ is a topological $n-1$ sphere. Hence $\pi_1(M) = 0$.

4.9. *Proof that $\pi_1(H, X) = 0$*. Using the Lemma (4.5), there is a neighborhood U of M such that every curve in U can be deformed into M. By (4.8), $\pi_1(M) = 0$, hence $\pi_1(U, X) = 0$.

Let γ be any closed curve in H. Then $\gamma' = h(\gamma)$ is a closed curve in $S^n - C$. Let E be a closed 2-cell and K a singular image of E in S^n such that γ' is the boundary of K. If $K \cap C = 0$, $h^{-1}(K)$ gives a singular disk in H that γ bounds. If $K \cap C \neq 0$, draw a finite number of disjoint, simple closed curves in E whose interiors contain the complete counterimages of $K \cap C$ and such that if $B_1, \cdots, B_k$ denote the images of these curves in K, $B_i \subset h(U)$, $i = 1, 2, \cdots, k$. Denote further the singular disk in K bounded by B_i by Q_i, then $K \cap C \subset Q_1 \cup \cdots \cup Q_k$. Since $h^{-1}(B_i) \subset U$ and $\pi_1(U, X) = 0$, there are singular disks in X such that $h^{-1}(B_i) \simeq 0$. By fitting these disks properly onto $h^{-1}(K - Q_1 - \cdots - Q_k)$, we find $\gamma \simeq 0$ in X, that is, $\pi_1(H, X) = 0$.

5. $i=n-2$. If the set C is a closed, topological $n-2$ cell in the n-sphere S^n, S^n-C fails to be 1-ULC. As above, if V, U are neighborhoods of C, the part residual to C is denoted by V_c, U_c, respectively.

5.1. LEMMA. *Given a neighborhood U of C there exists a neighborhood V of C such that any small path in V_c is null-homotopic in U_c.*

Let p be an interior point of the set C. Let V and W be neighborhoods of p in S^n such that $H_1(V_c, W_c)$ is infinite cyclic. For an edge point p let $W=U$ and choose V such that $H_1(V_c, W_c)$ is the null-element. This is possible from Corollary II, §2 of [1]. To each W let $Q\subset V$ be a neighborhood of p such that $C(Q, W)=0$, which is possible since C has the un.a.l.g. property. Thus if p is an interior point, $\pi_1(Q_c, W_c)$ is infinite cyclic and reduces to the null-element otherwise.

5.2. The sets Q just mentioned may be chosen as spherical neighborhoods, thus connected. Since dim $C=n-2$, each Q_c is connected.

5.3. Since C is compact, there is a finite covering of C by $Q^1, \cdots, Q^k$. We agree to retain in this covering at least one element, say Q^1, which corresponds to an *edge point* of C.

5.4. Let $\tau>0$ be such that any set of diameter $<\tau$ in $S(C, \tau)-C$ is contained in some Q^j_c. (If τ_2 is the Lebesque number of the covering $\{Q^1, \cdots, Q^k\}$ with respect to $\bar{S}(C, \tau_1)\subset Q^1\cup \cdots \cup Q^k$, let $\tau=\min(\tau_1, \tau_2)$.) Define the V of the lemma to be $S(C, \tau)$.

5.5. Let Q be one of the above neighborhoods corresponding to an interior point p of C. Let q be any other point in the intersection of Q and the interior of C. For $\delta>0$ define $Z=S(q, \delta)\cap Q$. Since $q^1(q, S^n-C)=1$, $H_1(Z_c, W_c)$ is infinite cyclic for δ sufficiently small [1]. Let $f\subset Q_c$ generate $H_1(Q_c, W_c)$ and $g\subset Z_c$ generate $H_1(Z_c, W_c)$. Since $Z_c\subset Q_c$, $g\subset Q_c$. Thus $g\sim f^k$ in W_c. The operation of bounding establishes a homomorphism of $H_1(Z_c, W_c)$ into $H_1(Q_c, W_c)$. To see this homomorphism is actually onto consider $H_1(Q_c\cup q; W_c\cup q)$ and $H_c(Q_c, W_c)$. Since the former reduces to the null-element, any element of the latter group, in particular f, bounds a chain in $W_c\cup q$. Since this chain does not lie in W_c, it contains q. The part of this chain that meets the surface of a small sphere about q is a 1-cycle that (bounding in W_c) may be expressed in terms of g. Thus f may be so expressed. Let $g^{k'}\sim f$. Combining this with $f^k\sim g$ gives $f\sim f^{kk'}$. Unless $|k|=|k'|=1$ this contradicts the fact that f is a generator of *free* group.

In view of $\pi_1(Z_c, W_c)\approx H_1(Z_c, W_c)$ and $\pi_1(Q_c, W_c)\approx H_1(Q_c, W_c)$, the above may be interpreted as follows: *if q is any interior point of C in Q, then in any sufficiently small neighborhood of q we may choose a path in S^n-C that generates $\pi_1(Q_c, W_c)$.*

5.6. To complete the proof of the above Lemma 5.1 let $k(S^1)$ be any closed path in V_c (5.4) of diameter $<\tau$. Suppose $k(S^1)\subset Q^{j_1}_c$. Let p_{j_1} be the point of C determining Q^{j_1} in the given covering. Let $T\subset C$ be an arc from

p_{j_1} to p_1. Without loss the interior of the arc may be assumed to lie in the interior of C. By 5.3, we see that the sets $Q^1, \cdots, Q^k$ cover C and that there is a chain $Q^{j_1}, \cdots, Q^{j_m}=Q'$ $(=Q^1)$ such that (a) $\bigcup_{i=1}^m Q^{j_i}\supset T$, (b) $Q^{j_i}\cap Q^{j_{i+1}} \supset a_{j_i}\in T\subset C$, $i=1, 2, \cdots, m-1$, and (c) $\pi_1(Q_c^{j_m}, W_c^{j_m})=0$. Let f_{j_1} be a generator of $\pi_1(Q_c^{j_1}, W_c^{j_1})$ that is contained in $Q_c^{j_1}\cap Q_c^{j_2}$. This is possible by 5.5. By 5.2 these sets are connected, hence the same base point may be used for k as for f_{j_1}.

If $\pi_1(Q_c^{j_1}, W_c^{j_1})$ reduces to the identity, the conclusion of the lemma is established. Otherwise $k\simeq f_{j_1}^{\lambda_1}$ in $W_c^{j_1}$. Let l_1 be an arc in $Q_c^{j_2}$ from the base of k to a point of f_{j_2}, where f_{j_2} is a generating element of $\pi_1(Q_c^{j_2}, W_c^{j_2})$ in $Q_c^{j_2}\cap Q_c^{j_3}$. Then, basing our paths at a point common to l_1 and f_{j_2}, we have, since $f_{j_1}\subset Q_c^{j_2}$, $l_1^{-1}f_{j_1}l_1\simeq f_{j_2}^{\lambda_2}$ in $W_c^{j_2}$. Hence $l_1^{-1}kl_1\simeq f_{j_2}^{\lambda_1\lambda_2}$ in $W_c^{j_1}\cup W_c^{j_2}$. Continuing in this manner, after a finite number of steps,

$$\overset{-1}{l}_{j_m-1}\cdots \overset{-1}{l_1} kl_1l_2\cdots l_{j_m-1}\simeq f_{j_m}^{\lambda_1\lambda_2\cdots\lambda_m}$$

in $W_c^{j_1}\cup\cdots\cup W_c^{j_m}$. But $\pi_1(Q_c^{j_m}, W_c^{j_m})=0$. Setting $l_1l_2\cdots l_{j_m-1},=x$ we obtain $x^{-1}kx\simeq 0$ in $W^{j_1}\cup\cdots\cup W^{j_m}$, that is, $k\simeq 0$ in U_c. This completes the proof of the Lemma 5.1.

5.7. Since S^n is compact, Lemma 5.1 shows that there is a $\delta>0$ such that any closed path in S^n-C of diameter $<\delta$ is $\simeq 0$ in $A=S^n-C$.

5.8. Let $f(S^1)$ be any closed path in S^n-C. Then $f\simeq 0$ in S^n. Let F be a mapping function realizing this homotopy, choosing the planar 2-cell E bounded by S^1 as range. If $F(E)\cap C=0$, no modifications are necessary. If $F(E)\cap C\neq 0$, proceed as follows. Since A is 0-ULC, to the δ of 5.7 there is a $\delta^0>0$ such that pairs of points in A whose distance apart is $<\delta^0$ can be joined in A by arcs of diameter $<\delta/2$. By the continuity of F there is a $d>0$ such that points in E having a distance apart less than d have images under F no further than δ^0 apart. Subdivide E now simplicially into cells of mesh $<d$. If $p\in S^1$, define $G(p)=F(p)$. Consider the 0-cells of E interior to S^1. If $F(p)\in S^n-C$, put $G(p)=F(p)$. If $F(p)\in C$, choose a point $q=G(p)$ near $F(p)$ but in S^n-C. Since C is nowhere dense in S^n, this may be done. In fact, if $\eta=\min\,[\delta^0-\max \text{ distance } F(p), F(p^1)]$, p, p^1 ranging over 0-cells of a fixed 2-simplex and min taken with respect to all 2-simplexes of E, then $q=G(p)\in S[F(p), \eta/3]$ gives a partial realization of G of mesh $<\delta^0$. Consider any 1-cell of E. By the way in which G has been defined on its ends and the significance of δ^0, G may be extended continuously over all of the 1-cell so that the image lies in A and has a diameter $<\delta/2$. Doing this simultaneously for all 1-cells we are ready to extend G to the interior of the 2-simplexes of E. Since the image of G on the boundary of any such 2-simplex has a diameter $<\delta$, G may be extended continuously into the interior so that the image lies in A. Since only a finite number of such cells are involved, the continuity of the deformation function G on E is clear. Thus $f(S^1)\simeq 0$ in A.

5.81. The above proof $(k=n-2)$ evidently holds under a slightly weakened hypothesis. Define $Y=(S^n-C)\cup(\text{interior } C)\cup(a)$, where a is any edge point of C. Assume that at each point of Y given a relative neighborhood U there is a relative neighborhood V such that $\tilde{\mathcal{C}}(V, U)=0$, then $\pi_{1_i}(S^n-C)=0$. As a special case we have for $n=3$:

COROLLARY. *Let C be a simple continuous arc in S^3 that is locally polygonal except for one end point $b(\neq a)$. Then $\pi_1(S^3-C)=0$.*

That the set Y must contain all interior points of C and at least one edge (end) point is indicated by the work of Fox-Artin, *Some wild cells and spheres in 3-space*, Ann. of Math. vol. 49 (1948) pp. 979–990. Example 1.1 shows that C must not be "twisted" near all boundary points of C.

5.9. $i<n-2$. By Corollary II, §2 of [1], S^n-C is 1-ulc. Since S^n-C has the un.a.l.g. property, it follows from Theorem I that S^n-C is 1-ULC. Since S^n-C is also 0-ULC, S^n-C is ULC^1. Since C is LC^1. Theorem 8a of [3] applies and we conclude $\pi_1(S^n-C)=0$.

6. In this section we show that if D is a compact, totally disconnected set in S^n such that S^n-D has uniformly abelian local fundamental groups, then $\pi_1(S^n-D)=0$. Examples due to Antoine show that this need not be the case in general [2].

6.1. THEOREM V. *Let D be a compact, totally disconnected subset of S^n. If S^n-D has uniformly abelian local fundamental groups, then S^n-D is* 1-ULC.

Proof. Define $A=S^n-D$. Then A is open and by Theorem I it is sufficient to show A is 1-ulc over the integers. By Theorem 1 of [7], choosing $M=D$ above, and noting the Pontrjagin extension of the duality formula, A is 1-ulc over the integers.

6.2. THEOREM VI. *If D is a compact, totally disconnected subset of S^n, and S^n-D has uniformly abelian local fundamental groups, then $\pi_1(S^n-D)=0$.*

Proof. Let $f(S^1)=K$ be any continuous 1-sphere in S^n-D. In discussing homotopies of f we choose a closed 2-cell E^2 as range instead of $S^1\times I$. Let E^2 have polar coordinates r, θ. Put $t=1-r$ and let $F(\theta, t)$ be any deformation of f to a point in S^n. $F(\theta, 0)=f(\theta)$, $F(\theta, 1)=\text{constant}$.

Consider $H=F^{-1}(D)$, that is, H is the totality of points in E^2 carried into D by F. If H is null, we already have the desired homotopy.

6.3. If H is totally disconnected, proceed as follows. Choose $\epsilon=1$, then by Theorem V there is a δ such that all paths in S^n-D of diameter $<\delta$ are $\simeq 0$ in S^n-D. To $\delta>0$ there is a $d>0$ such that any set of diameter $<d$ in E^2 has an image of diameter $<\delta$. Since H is 0-dimensional, compact and lies in the interior of E^2, we may draw in E^2-S^1 a finite number of disjoint simple closed curves of diameter $<d$, $B^1, \cdots, B^k$, whose interiors are disjoint and contain H. We now define a new deformation of f to a point as follows. If a point of E^2 is exterior to each of the simple closed curves, $G(\theta, t)=F(\theta, t)$.

On the interior of one of the curves B^i we use the fact that F on this simple closed curve defines a path in $S^n - D$ of diameter $<\delta$, hence G, on the interior of B^i, may be continuously extended so that the image lies in $S^n - D$. In this manner $G(\theta, t)$ is defined continuously over all of E^2 and represents a contraction of f to a point in $S^n - D$.

6.4. If H is not totally disconnected, proceed as follows. Let (H_α) be the collection of components of H. A new mapping will be defined by replacing this collection by that obtained by coalescing certain elements. If $H_{\alpha'}$ is an element of (H_α) that does not separate E^2 and is not contained in a domain complementary to an H_β which does separate E^2, we associate $H_{\alpha'}$ with itself. If $H_{\alpha'}$ does separate E^2 or if it is contained in a domain complementary to an H_β which does, then we associate $H_{\alpha'}$ with the maximal H_β with this property. Let the new collection of elements be $(\tilde{H}_\alpha)$. This collection is upper semi-continuous. A new mapping G is now defined on E^2 as follows: If $p \in E^2 - H$, $G(p) = F(p)$. If $p \in H$, $G(p) = F(q)$, where p determines H_α and q is any point of the corresponding element in $(\tilde{H}_\alpha)$. We note G agrees with f on S^1 and G performs a contraction of f to a point in S^n. Apply the Eilenberg-Whyburn factor theorem [6] to G, obtaining a monotone transformation G_m on E^2 followed by a light transformation G_l on $Z = G_m(E^2)$. Since no inverse set of G_m separates E^2, Z is a 2-cell [4]. But then G_l is topologically equivalent to the type of transformation considered above when H is totally disconnected (6.3).

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